Unedited Letters from Euler to d'Alembert

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The following letters belong to d'Alembert's papers that Mrs. O'Connor, the daughter of Condorcet, left to the library at the Institut de France. With d'Alembert's letters to Turgot which we published in *Correspondance inédite de d'Alembert*, they are part of the bundle denoted R $69g^6$ in-4°.

They concern analysis and astronomy, primarily the question of logarithms of negative numbers. Euler, like Leibniz, maintained that negative numbers do not have real logarithms, whereas d'Alembert, like Bernoulli, claimed the opposite. Euler resolved the problem by bringing it back to logarithms of circular and exponential functions. One reads with the utmost interest the profound discussion of Euler. There is no need to emphasize this now classic solution.

The first letter is dated April 1747, the last has no date, but the mention of the publication of the treatise in *Science Navale* that it contains allows us to date it to 1749. Equally interesting, it allows us to specify the date at which the treatise was finished; it was 1741 when Euler posed a problem on the hydraulic tourniquet which has since been resolved and applied to devices named *reaction wheels*.

These letters are evidently only fragments of a much greater correspondance; they should be considered as a Supplement to the Mathematical Correspondance published in 2 volumes by Fuss in 1843, to the *Opera Minora* of Euler (1849) and to the *Opera posthuma* (1862).

Euler à d'Alembert [1]

Dear sir,

It is quite true that the example of the curve $y = \sqrt{x} + \sqrt{x}\sqrt{x+a}$, which when $a = 0$ suddenly loses one half, does not prove that the same thing should happen with the curve $y = \frac{1}{n-1} - \frac{1}{(n-1)(x^{n-1}-1)}$ when $n = 1$; further, I did not use this example except to show the possibility of such a disapperance in a certain case, and I only draw the conclusion that, although the latter curve always has a diameter when n is an odd number, still these results could perhaps stop being true when $n = 1$. By this means, it seems to me that I have responded well to your objection, drawn from this general formula, although this case proves nothing for my thesis, because at first I intended to show that the arguments that are claimed to prove the realness of the logarithms of negative numbers are not too sure. But it seems to me that my theory does not lack positive proofs, but before I can spread them out, I must respond to your objection, based on the

[1] This letter bears the address To Sir, Mr. D'Alembert, Member of the Royal Academy of Sciences in Paris and Berlin at Paris

equation $y = e^x$, where you think that the number e can have either a positive or a negative value. I admit that even the value is completely arbitrary, because if you let $e = 10$, the exponent x will be the common or tabular logarithm of the number y and if $e = 2,305$ etc. or $e = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdots 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4}$ etc., x will be the hyperbolic logarithm of the number y. But as soon as e is assigned a definite value, the entire system of logarithms of all numbers is determined, as well as the curve with equation $y = e^x$, and as e is almost its parameter, one cannot give it at the same time two different values that the curve is not composed of two different curves. In the same way the parabolic equation $y^2 = ax$, if a is given a double value like $a = +1$ and $a = -1$, one has two different curves, which are not joined by the line of continuity. That said, it seems quite clear to me that letting $e = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdots 2\cdot 3}$ etc. the logarithms of negative numbers should be impossible, given that it is impossible to find a value of x such that e^x or $1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3}$ produces a negative number. It seems paradoxical to you that the differentials of $\ln(y)$ and $\ln(-y)$ are the same; but you will grant me still this equality in a more general sense, that is to say that $\frac{dy}{dx}\ln(y) = \frac{dy}{dx}\ln(ay)$, where a is some constant number, hence I don't see the least difficulty why we can reject the case where $a = -1$. By the reasoning that you prove that $\ln(-1) = 0$, you prove as well that $\ln(\sqrt{-1}) = 0$, because since $\sqrt{-1}\cdot\sqrt{-1} = -1$, you will have $\ln(\sqrt{-1}) + \ln(\sqrt{-1}) = \ln(-1)$, that is to say $2\ln(\sqrt{-1}) = \ln(-1) = \frac{1}{2}\ln(+1)$, which leaves $\ln(\sqrt{-1}) = \frac{1}{4}\ln(1) = 0$, and if you don't agree with this reasoning, you will grant me that the first is no longer convincing. Now, you must at least agree that the logarithms of imaginary numbers are not real, otherwise the expression $\frac{\ln(\sqrt{-1})}{\sqrt{-1}}$ is known to express the squaring of the circle. Let $\frac{\ln(\sqrt{-1})}{\sqrt{-1}} = \alpha$ and you have $\ln(\sqrt{-1}) = \alpha\sqrt{-1}$, which is an imaginary number. Then if $\ln(\sqrt{-1})$ is imaginary, why wouldn't $2\ln(\sqrt{-1}) = \ln(-1)$? Then as $(\frac{\sqrt{-1}+\sqrt{-3}}{2})^3 = 1$, following your reasoning, you would have $3\ln(\frac{-1+\sqrt{-3}}{2}) = \ln(1) = 0$ and the logarithm of $\frac{-1+\sqrt{-3}}{2}$ would be as equal to 0 as $\ln(+1)$ and $\ln(-1)$ and $\ln(\sqrt{-1})$ etc., which is not tenable. But you contest that even $\ln(1)$ must be imaginary being $2\ln(-1) = 4\ln(\sqrt{-1}) = 3\ln(\frac{-1+\sqrt{-3}}{2}$ etc. Well, that is exactly what I want, since I said that $\ln(+1)$ has infinitely many different values among which there is one equally 0 and all the others are imaginary. To better explain this let $0, \alpha, \beta, \gamma, \delta\varepsilon, \zeta, \eta, \theta, \iota etc.$ be the logarithms of the unity and I say that the values of $\ln(-1)$ will be $\frac{\alpha}{2}; \frac{\gamma}{2}; \frac{\delta}{2}, \frac{\zeta}{2}$ etc. all imaginary, yet such that the double of each is found among the logarithms of $+1$; but it does not follow that half of each of the values of $\ln(1)$ are found among $\ln(-1)$, since -1 is just one value of $\sqrt{+1}$, the other being $+1$, thus the logarithms are $\frac{0}{2}, \frac{\beta}{2}, \frac{\delta}{2}, \frac{\zeta}{2}$ which are exactly the same as $0, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, etc. Because $\frac{\beta}{2} = \alpha, \frac{\delta}{2} = \beta, \frac{\zeta}{2} = \gamma, \frac{\theta}{2} = \delta$, etc. In the same way as the three cubic roots of 1 are 1, $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$ the logarithms of these three roots will be

$$\ln(1) = \frac{0}{3}, \frac{\gamma}{3}, \frac{\zeta}{3}, \frac{\iota}{3}, \frac{\mu}{3}\text{etc.}$$

the same as $0, \alpha, \beta, \gamma, \delta, \varepsilon$, etc.

$$\ln(\frac{-1+\sqrt{-3}}{2}) = \frac{\alpha}{3}, \frac{\delta}{3}, \frac{\eta}{3}, \frac{\chi}{3}, \frac{\nu}{3}\text{etc.}$$

$$\ln(\frac{-1-\sqrt{-3}}{2}) = \frac{\beta}{3}, \frac{\varepsilon}{3}, \frac{\theta}{3}, \frac{\lambda}{3}, \frac{\xi}{3}\text{etc.}$$

and the letters $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. are not fonded on pure conjecture; I have the honor to even show you their actual values. For, let π be the circumference of a circle, which has radius =1 and the values of $\ln(+1)$ are $a \pm \pi\sqrt{-1}$; $\pm 2\pi\sqrt{-1}$; $\pm 3\pi\sqrt{-1}$; $\pm 4\pi\sqrt{-1}$; $\pm 5\pi\sqrt{-1}$; etc. and of $\ln(-1)$ are $\pm\frac{1}{2}\pi\sqrt{-1}$; $\pm\frac{3}{2}\pi\sqrt{-1}$; $\pm\frac{5}{2}\pi\sqrt{-1}$; etc.

And in general I found that $\ln(1^p) = \pi(mp+n)\sqrt{-1}$, $\ln(-1)^p = \pi(\frac{1}{2}p + mp + n)\sqrt{-1}$; where m and n represent integers that can be positive or negative. By this means, all difficulties in logarithms of negative, based on $2\ln(-1) = \ln(+1) = 0$, since by the same reasoning we would be obliged to say that $\ln(\sqrt{-1}) = 0$ and $\ln(\frac{-1+\sqrt{-3}}{2} = 0$, fully disappear.

You say again, good sir, that since $e^x = y$, if $x = \frac{1}{2}$, the number y can be positive or negative; but since e^x denotes here the value of the series $1 + \frac{x}{1} + \frac{x^2}{2\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} +$ etc. I believe that I have responded very firmly that e^x never has more than one value, and that is positive, still x would be a fraction where the calculation of the root seems to make the formula e^x equivalent.

Your piece on the movement of the moon is without doubt of the greatest depth, and your superiority with the most difficult calculations shines throughout it. The comment that I took the liberty to write you only concerns the application of your analysis for the use of astronomical tables. To this end, it is necessary to have easy approximations for the calculations and it seems to me that the manner in which you treat this problem is not too neat with regard to these approximations. As I have been handling this question in a number of different approaches, I have found only one way which was fit for astronomical usage, with which I also calcualted my lunar tables. I am thus even more curious to see the rest of your resarch on this matter. having the honor to be with the highest regard,

Berlin, this April 15 1747.

Your very humble and very obedient servant,

L. Euler.

Euler à d'Alembert [2]

Dear sir,

I take advantage of the departure of Mr. Delisle to respond to your last letter and to reccomend to you a young man from our Academy, who has obtained the permission to accompany Mr. Delisle. He is the son of one of our astronomers, named Grischow who, having made some progress in astronomy, believes he cannot use his time better than to find an opportunity to benefit from the insight and direction of the Astronomers of Paris, to whom I beg you to allow him access and to particularly honor him with your welcome.

For our controversy concerning the logarithms of negative and imaginary numbers, I hope that it will soon finished; in your paper on integrals, which was just printed in the second volume of our dissertations, I followed your orders crossing out the article where you spoke of the logarithm of -1 and I believe that you will soon be entirely in agreement with me on this subject. I confess that the formula e^x should have two values in the case when $x = \frac{1}{2}$; but you

[2]This letter bears the address To Sir, Mr. D'Alembert, Member of the Royal Academy of Sciences in Paris and Berlin at Paris.

will grant me too that in other cases the value of e^x cannot be negative and since it is prirmarily a question of the logarithm of -1, you won't claim that e^x could be -1, supposing that $x = 0$, then at least this argument proves nothing for you.

When you say that we could resolve $\ln(-x)$ with a series whose value became real, I understand nothing, if it's not the terms of the series are real; but even $\sqrt{-x}$ could be resolved with such a series. Furthermore, I agree that what I said in my first letter on the series $e^x = 1 + x + \sqrt{x}2+$ etc. proves nothing for me, as well as the ambiguity of e^x in some cases proves nothing against me either, since we should also grant three values when $x = \frac{1}{3}$, four when $x = \frac{1}{4}$ etc., but this would lead us too far.

When you say that the number e should not be considered as a paremeter of the logarithm, but as the coordinate which corresponds to the abscissa $x = 1$ and it is because of this that it can be have a positive value just as it can have a negative one, I could say with just as much right that logarithms have not only two solutions equal and similar according to the two formulas $x = \ln(y)$ and $x = \ln(-y)$, but also as many as one would like $x = \ln(y), x = \ln(my), x = \ln(ny)$ etc. since all these formulas have the same derivative $dx = \frac{dy}{y}$. For in regards to your transformation of e^x by $\frac{e^{\frac{x}{g}}}{a^{\frac{x}{g}-1}}$ where x is to g as an odd number to an even one, we could with just as much right imagine this formula with $x : g =$ even: odd or odd: odd and thus you wouldn't find your tale. It seems to me that all these reasons are not strong enough to prove that $\ln(x) = \ln(-x)$.

Next you doubt if the formula draw from sine gives all the logarithms of -1; but

I don't know if a simple doubt dismissed by the proof can overthrow what I set up and for the formula $\frac{\ln(\sqrt{-1})}{\sqrt{-1}}$ I maintain that it does not contain any values other than $\frac{(4n+1)\pi}{2}$, n marking an arbitary integer and π the circumference of a circle having diameter $= 1$, so this formula could never become $= 0$. It is true that my opinion is applied on the formula drawn from the sine, but I don't see any reason why this formula gives all the logarithms of $\sqrt{-1}$ and I still believe that the reasons for are stronger than the ones against.

At last in the formula of the circular arc $s\sqrt{-1}\ln(x + \sqrt{x^2-1})$ if x marks the cosine of the arc s, and I don't see any reason to doute, that if $x > 1$, the arc s is not a simple imaginary number $b\sqrt{-1}$, such that

$$\ln(x + \sqrt{-1}) = b;$$

and I don't believe that you will prove the opposite.

I passed on to the Academy a paper on this subject, where I believe I have so brought up to date this matter, that at least to me, I don't find the least difficulty anymore, although previously I had been extremely embarrased.

Sir, I have the honor to be your obedient servant,

L. Euler

P.S. You will grant me that $\ln(+1) = \pm 2n\pi\sqrt{-1}$ and that $\ln(-1) = \pm(2n - 1)\pi\sqrt{-1}$ but you say, good sir, that among the logarithms of -1 is also found 0; then, since two logarithims of -1 added together give $\ln(+1)$, the logarithms of $+1$ will be not only $\pm 2n\pi\sqrt{-1}$ but also $\pm(2n-1)\pi\sqrt{-1}$. Further, you will grant me that $\ln -1 = \pm frac4n \pm 1)2\pi\sqrt{-1}$ and that $\ln(-\sqrt{-1} = \pm\frac{(4n\pm1)}{2}\pi\sqrt{-1}$, but that these formulas don't contain all the logarithms of $+\sqrt{-1}$ and of $-\sqrt{-1}$, and that they contain 0 as well, so since $\ln(+\sqrt{-1} + \ln -\sqrt{-1} = \ln(+1)$, the logarithm of 1 includes as well the formulas $\pm\frac{(4n\pm1)}{2}\pi\sqrt{-1}$. Likewise, if you say that zero is also the logartihm of the highest imaginary roots of 1, you would at least be obligated to say that all the logarithms of +1 are contained in the formula $\frac{m}{n}\pi\sqrt{-1}$ where $a\sqrt{-1}$, whatever quantity that we take for a; so that $\ln(+1$ would become completely indeterminate, a consequence that seems to be sufficient to destroy your objection. But, following my point of view, when I say that:

$$\ln(+1) = 0; \pm2\pi\sqrt{-1}; \pm4\pi\sqrt{-1}; \pm6\pi\sqrt{-1};\ \text{etc}$$

$$\ln(+\sqrt{-1}) = +\frac{1}{2}\pi\sqrt{-1}; +\frac{5}{2}\pi\sqrt{-1}; +\frac{9}{2}\pi\sqrt{-1};\ \text{etc.}$$

$$-\frac{3}{2}\pi\sqrt{-1}; -\frac{7}{2}\pi\sqrt{-1}; -\frac{11}{2}\pi\sqrt{-1}\ \text{etc.}$$

$$\ln(-1) = \pm\pi\sqrt{-1}; \pm3\pi\sqrt{-2}; \pm5\pi\sqrt{-1}; \pm7\pi\sqrt{-1};\ \text{etc.}$$

$$\ln(-\sqrt{-1}) = +\frac{3}{2}\pi\sqrt{-1}; +\frac{7}{2}\pi\sqrt{-1}; +\frac{11}{2}\pi\sqrt{-1};\ \text{etc.}$$

$$-\frac{1}{2}\pi\sqrt{-1}; -\frac{5}{2}\pi\sqrt{-1}; -\frac{9}{2}\pi\sqrt{-1}$$

You will find the most beautiful harmony; because two arbitrary logarithms -1 added together will always produce a $\ln(+1)$; two logarithms of $+\sqrt{-1}$ added together will always give a $\ln(-1)$; in the same way that two logarithms of $-\sqrt{-1}$ and a $\ln(+\sqrt{-1}+$ a $\ln(-\sqrt{-1}$ will always give a $\ln(+1)$. This remark alone seems to me sufficient to convince you of the truth of my position, as a matter of fact, if you make the smallest change in my formulas, you would be obligated to make the logarithms of +1 completely indeterminate; and I ask you to weight well this argument.

Euler à d'Alembert [3]

Dear Sir,

Having learned from Monsieur de Mauperterius that you wish to abandon for some time your mathematical research to reestablish your health, which has found itself considerably weakend by your overzealous diligence, I so strongly approve of this resolution, wishing you every success that you are due, that I do not want to trouble you with reflections on imaginary logarithms, besides I wouldn't know what I could add on this matter, that I have not already pointed out, and I highly doubt, if my work on this matter would be able to alleviate all your doubts, that you have gone to such trouble to propose to me. But after you have have agreed with me this much, these doubts don't support your point of view, and there is no one who would know how to better resolve them than yourself.

If for your amusement you want to do some research that doesn't require a lot of effort, I will take the liberty to propose the expression $(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc., which expanded by the current multiplication, gives the series $1-x^1-x^2+x^5+x^7+-x^{12}-x^{15}+x^{22}+x^{25}-x^{35}-x^{40}+x^{51}+x^{57}-x^{70}-x^{77}+$ etc., which seems to be very remarkable, because of the law that can be easily discovered; but I don't see how this law can be deduced without induction of

If we let $s=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc., I can prove that there will be $s=1-\frac{x}{1-x}+\frac{x^3}{(1-x)(1-x^2)}-\frac{x^6}{(1-x)(1-x^2)(1-x^3)}+\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}-$ etc.

[3]This letter bears the address To Sir, Mr. D'Alembert, Member of the Royal Academy of Sciences in Paris and Berlin at Paris.

the proposed expression itself.

I have the honor to be with the most perfect consideration, thanking you for all the kindness that you have for our Monsieur Grischow,

Sir, your very humble and very obedient servant,

L. Euler

Berlin, this April 30 December 1747.